Journal of Statistical Physics, Vol. 55, Nos. 5/6, 1989

Book Review: *Random Media*

Random Media. G. Papanicolaou, ed. The IMA Volumes in Mathematics and Its Applications 7. Springer, New York, 1987.

This volume is the proceedings of a workshop on random media (September 18–24, 1985) that was held at the Institute for Mathematics and Its Applications (University of Minnesota, Minneapolis) as part of the 1985–86 program on Stochastic Differential Equations and Their Applications.

Volume 7 contains 21 papers in the general area of random media, both theoretical and applied. Some describe new results, some are reviews of earlier work that has appeared elsewhere. A glance through the table of contents shows that a rather broad range of topics is covered: transport properties of random media (e.g., diffusion, wave propagation, conductivity), random Ising models, random Schrödinger operators, amorphous semiconductors, numerical algorithms for SDEs, etc. It would take up too much space to cover each paper, so I shall make a (somewhat arbitrary) selection.

Baxter and Jain (1; see the end of this review for a list of papers) study Brownian motion B_t in R^d $(d \geqslant 3)$ that is stopped when it hits a closed set D_n. This set consists of many small sets, which become ever smaller and ever more numerous as n tends to infinity, in such a way that they evenly fill out the space with a limiting local density. If τ_n denotes the first hitting time of D_n, then it is shown that τ_n converges to a limit T (in some weak sense). This limit is characterized as an exponential killing time, i.e., $P(T>t)=\exp[-\int_0^t h(B_s)\,ds]$, where h is a simple function of the limiting density of D_n. This result translates directly into convergence for solutions of the diffusion equation on D_n^c with Dirichlet boundary conditions. It is also possible to add a time-dependent drift to B_t.

Carmona (4) gives a review of some spectral properties of one-dimensional random Schrödinger operators and perturbations of these by nonrandom potentials. His results concern existence, smoothness, localization,

0022-4715/89/0600-1327$06.00/0

and stability of solutions under very general conditions. For each of these a clear proof is included, which makes the paper self-contained and expository. In a short last section these results are applied to spherically symmetric random potentials in higher dimension to obtain localization.

Durrett (7) discusses the problem of a random walk in a random potential on Z^d. His main theorem, which generalizes earlier work by Sinai for $d=1$, states that if the potential has a certain natural scaling property, then for any $d \geqslant 1$ the process is recurrent and moves out to infinity logarithmically slowly, thus exhibiting strong subdiffusive behavior. Unfortunately, the conditions on the random potential rule out an independent medium as soon as $d>1$ (i.e., where the transition rates between nearest neighbors are independent). So the question of recurrence vs transience remains open for this "classical" case. A second theorem, valid for general media, leads to a necessary (but most likely not sufficient) condition for recurrence, and to a (strong) sufficient condition for transience. However, these are both formulated in terms of the stationary measure and therefore are difficult to handle in practice.

Imbrie (10) outlines his proof that the Ising model in a random magnetic field exhibits long-range order (=spontaneous magnetization) at zero temperature for $d \geqslant 3$ when the field is symmetric and sufficiently small. This results supports the (still unproven) conjecture that the lower critical dimension is 2 and rules out earlier speculations as to it being 3. The paper gives a very clear sketch of the key ideas behind the proof, which are based on subtle contour arguments and on large-deviation and entropy estimates for large contours. There are also some hints at why long-range order should persist for (small) positive temperatures, where a proof is still lacking.

Petersen (15) treats the problem of how to simulate solutions of the Ito SDE $dX = b(X)\,dt + \sigma(X)\,dW(t)$ on a computer [$W(t)$ is standard Brownian motion]. The discretization and iteration scheme requires a subtle choice of algorithm in order to avoid inaccuracy and instability. This is illustrated via examples. For instance, it is demonstrated that a "Runge-Kutta" algorithm of the type $X(t+h) = X(t) + hb(Y) + h^{1/2}\sigma(Y)\,Z$ ($h \ll 1$, Z standard normal), with Y chosen randomly to "interpolate" between $X(t)$ and $X(t+h)$, leads to desired results under certain restrictions on the functions b and σ. This paper focuses solely on simulation techniques and makes no attempt at a theory.

Soner (18) studies a sequence of N-dimensional diffusion processes $Y_N(r)$, $N=1, 2,\ldots$, that arise from the propagation of sound pressure waves in a random ocean. This propagation is characterized by "trapped modes" and "radiation modes." For a fixed number N of trapped modes, $Y_N(r)$ describes the energies associated with these modes as a function of (scaled) radial distance r from the source (and is itself an approximation valid for large radial distances and small fluctuations in the index of refraction). The two main theorems are a law of large numbers and a central limit theorem for $Y_N(r)$ as N tends to infinity. The limit process is characterized explicitly.

Below is a list of all the papers. This volume is a very useful overview of (some of) the mathematical theory of random media and may serve as a guide to the vast literature that has developed around this area over the past 10 years. For physicists and chemists it certainly contains many papers worth looking at, but I am afraid that the mathematical language may scare them off.

1. Stable convergence and asymptotic capacity measures
   J. R. Baxter and N. C. Jain
2. Effective equation and renormalization for a nonlinear wave problem with a random source
   L. L. Bonilla
3. The spectrum of backscatter for pulse reflection from a randomly stratified medium
   R. Burridge, G. Papanicolaou, and B. White
4. Exercises for a book on random potentials
   R. Carmona
5. Travel-time problems for waves in random media
   P. L. Chow
6. Elements of the theory of amorphous semiconductors
   M. H. Cohen
7. Multidimensional RWRE with subclassical limiting behavior
   R. Durrett
8. Localization for a random discrete wave equation
   W. G. Faris
9. Simulations and global optimization
   B. Gidas
10. Low temperature behavior in random Ising models
    J. Z. Imbrie
11. Some recent results on wave equations, path integrals, and semi-classical approximations
    J. R. Klauder
12. Effective conductivities of reciprocal media
    J. B. Keller
13. Reflection from a lossy, one-dimensional ocean sediment model
    W. E. Kohler
14. Problems of one-dimensional random Schrödinger operators
    S. Kotani

15. Numerical simulation of Ito stochastic differential equations on supercomputers
W. P. Petersen
16. Lyapunov exponent and rotation number of the stochastic harmonic oscillator
M. A. Pinsky
17. Regularity of the density of states for stochastic Jacobi matrices: a review
B. Simon
18. Limit theorems for measure-valued diffusion processes that arise in wave propagation in a randomly inhomogeneous ocean
H. M. Soner
19. Spectral properties of discrete and continuous random Schrödinger operators: a review
B. Souillard
20. Dispersive bulk parameters for coherent propagation in correlated random distributions
V. Twersky
21. Random rays and stochastic caustics
B. J. White

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